

The Boy Surface as Architecture and Sculpture

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I will discuss the architectural structure of a folly in the Arpam project generated by the Boy surface. The classic way to build the Boy surface will be recalled with a few qualitative observations. I will also discuss four sculptures related to the Boy surface that illustrate various aspects of knot theory.

Keywords : Boy surface, fiber space, trefoil, truncated polyhedron

AMS subject classification : 53A05, 55R05

1 Introduction

According to [14], Jean-Pierre Petit had suggested to the French artist Max Sauze a sculpture of the Boy surface. Petit's observations have been developed and expressed in fine mathematical terms by François Apéry [1] [2]. From the equations he established, François has produced several shining wire models of the Boy surface (a photo of the first one can be viewed in [3] ; we can look here at a photo of the most recent one). Apéry's student in Cagliari, Gregorio Franzoni, has made a similar model in Teflon. Several beautiful classical views of the Boy surface can be seen on Internet. Wikipedia gives a recent (2008) good animation showing its construction, it seems to be inspired by the static visual construction given in [17] (via the cross cap and the Roman Steiner surface); this web site also shows a wire sculpture in Oberwolfach. Classical visual and mathematical presentations of the Boy surface can also be found in the nice books by Anatoli Fomenko [8] and George Francis [9]. Polyhedral constructions of the surface have also recently been made by Ulrich Brem [5], Richard Denner [7], Rob Kirby [12], Konrad Polthier through an animation [16].

In this article, for the readers not familiar with this surface, I first recall the classical way to build it - to attach a 3-Möbius band to a 2-disk - then add a very few qualitative observations.

In the late sixties the Boy surface was used to obtain an eversion process for the 2-sphere. I will use the Boy surface to design an architectural project associated with the Arpam project [4].

Recall that the Arpam project proposes to construct several small buildings, referred to as follies, in a large verdant park. Each folly will illustrate mathematical concepts of a particular chapter of the mathematical universe.

The Boy surface will be used to generate the structure of a folly. Four knotted sculptures will be placed in the vicinity of the folly, where one knot sculpture will be related to the Boy surface.

2 What is the Boy surface

2.1 Notion of projective space

Light has played a fundamental role in the evolution of our world. Sunlight has brought the energy which has permitted the birth and the unfolding of the life. Through the perception and the vision of objects and their shadows, it has allowed the creation and the development of classical geometry.

If one considers a beam of parallel rays of light which projects the shadows of objects on the ground, its most primary and significant characterization is the direction of these rays. It can be considered as an abstract line in space. This direction varies during the day according to the physical source we have in the hand.

The set of all these directions of light in a given milieu according to which one can produce shadows is called a *projective space* attached to this milieu.

The projective space attached to our usual space \mathbf{R}^3 is usually denoted by \mathbf{RP}^2 or \mathbf{P}^2 .

2.2 A class of representations of \mathbf{P}^2

Let us consider a 2-sphere as a representation of the canopy. Any line through the centre of the sphere represents a ray, a direction of light coming from a star considered as a punctual source of light. This line crosses the sphere in two antipodal points (symmetrical points with respect to the centre of the sphere). We identify these two points as one point and the set of these identified points is a representation of projective space.

If we remove the equator, the sphere is divided into an upper and a lower open hemisphere (Figure 1). By pushing them on a plane, we get that these upper and lower parts of the sphere are topologically equivalent to an open 2-disk, that is a disk without the circle which bounds it. In that case, the identification of the antipodal points of these two parts is immediate : we get a surface which can be one of these parts, an open 2-disk.

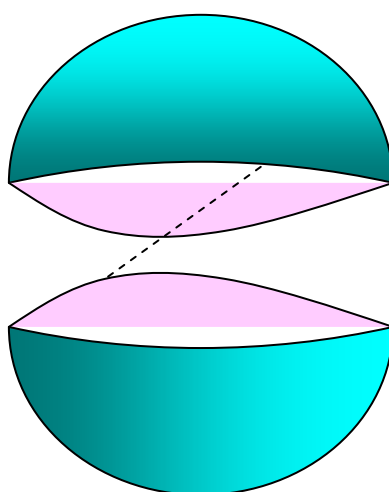


Figure 1

But now we have to deal with the equator. After a while of unsuccessful search, you may remember a previous process of identification which you can view as a process of

identification of antipodal points : the construction of a Möbius band. Take a small strip of antipodal points on the sphere and any two antipodal points P and P' (Figure 2). Cut the strip in P' say (Figure 3), and make an half turn of the strip (Figure 4) so that P and P' are glued into one point : one gets the Möbius band (Figure 5).

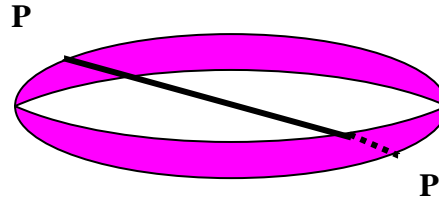


Figure 2 : Two antipodal points on the circular strip around the equator

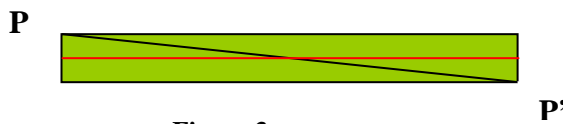


Figure 3



Figure 4

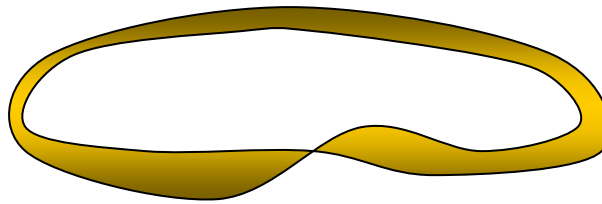


Figure 5 : The classical 1-Möbius band got by an half-turn of the original strip

Now we can get a representation of \mathbf{P}^2 . Divide the sphere into three parts : a small strip along the equator forming a neighbourhood of this equator, the remaining upper and lower parts of the sphere. The circular strip is bounded by two symmetric circles belonging to the strip, an upper one and a lower one. The upper and lower parts of the sphere are also bound, each one, by a similar circle. By glueing the upper and lower parts of the sphere to the strip along their common upper and lower boundaries, the sphere is build again.

But if we identify the antipodal points of the so divided sphere, we get a disk and a Möbius band together with their boundaries. If we attach the two surfaces by gluing their boundaries, we get a complete representation of the projective space.

In the previous construction of the 1-Möbius band, the initial strip had been half turned only one time (Figure 6). But let us remark that if one makes three half-turns, more generally an odd number $(2p + 1)$ of half-turns, the points P and P' can always be identified, which is impossible if this number of half-turns is even (Figure 7). Thus we get the $(2p + 1)$ -Möbius band.



Figure 6

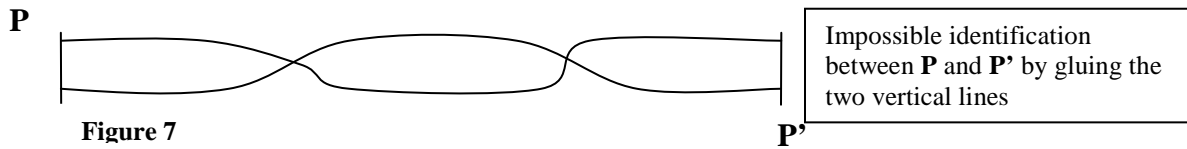


Figure 7

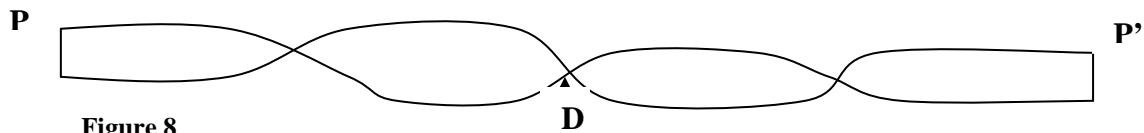


Figure 8

The 3-Möbius band is a two times covering of a 1-Möbius band

The original Boy surface, the *3-Boy surface*, was created in 1901 by using the 3-Möbius band (Figure 8). If the strip had been turned $(2p+1)$ times, we would have got a “ $(2p+1)$ -Boy surface”. It could have been named a “ $(2p+1)$ -Möbius surface” since, in 1874, Möbius created the first representation of \mathbf{P}^2 using a 5-Möbius band over a cone. But almost thirty years later, contradicting a conjecture by Hilbert, Boy showed that his surface is an immersion of \mathbf{P}^2 into \mathbf{R}^3 .

Let us look at the boundary of a 3-Möbius band (Figure 9, right) :

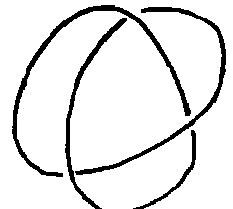
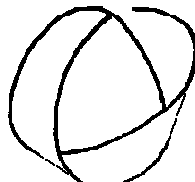
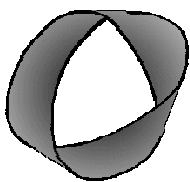


Figure 9 : The boundary of a 3-Möbius band (right)

We see that it is a trefoil knot. As the following picture shows (Figure 10), this knot can be imbedded into \mathbf{R}^3 in such a way that the three leaves are in general position (here each leaf belongs to a plane of an orthogonal frame of \mathbf{R}^3).

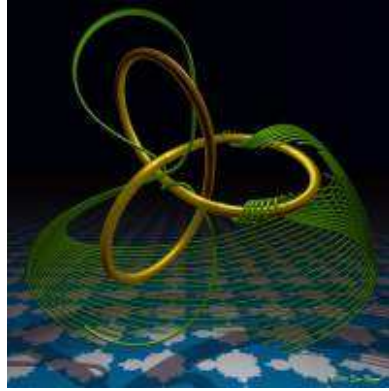


Figure 10 : Real Matrix 1, Jos Leys, 50 x50, 2006

Now let us go back to the construction of the 3-Boy surface. Consider first the 3-Möbius band forming the neighbourhood of the equator. We can give it a shape following the trefoil contour which we can see on the image. As the size of the neighbourhood decreases, the torsion of the leaves is getting lower, so that in the final state, the trefoil becomes the boundary of a 3-propeller curve (Figure 11) whose three fans are topological circles, each one being located in an orthogonal plane of the frame of the space. They meet in one singular point, the origin of the frame. This propeller curve is a representation of the equator after identification of its antipodal points. We have thus to give to our 3-Möbius band a shape taking into account the fact that its “axis”, the equator, is a three propeller curve. Now we can understand the structure of our 3-Möbius band : it is a surface foliated by standard trefoils except an exceptional one, the propeller.

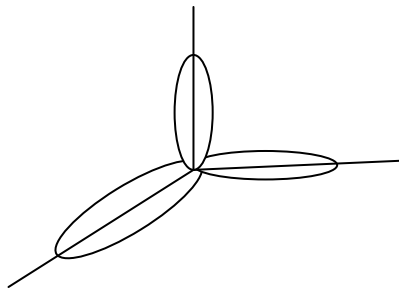


Figure 11 :The three-propeller curve

Then we can glue the boundary of this 3-Möbius band to the boundary of the 2-disk. Take any meridian on the sphere : it passes through the north pole of the sphere, and reaches two antipodal points of the equator. It is thus converted into a curve on the surface that passes two times through the point of that surface representing the two antipodal points of the equator.

The propeller curve is thus a self-intersection curve of the 3-Boy surface, to which it can be given a nice geometric shape. Here are a few graphic works made the French team :

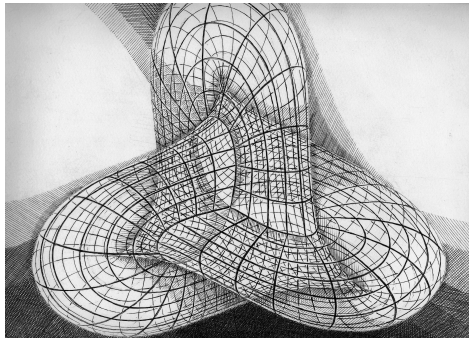


Figure 12
Surface de Boy
 Patrice Jeener, 33x25, 2003

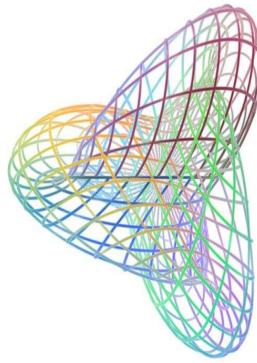


Figure 13
The Boy surface
 J.-F. Colonna, 2002, screen scale

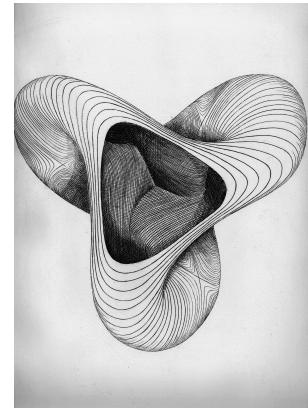


Figure 14
Ecorché de la surface de Boy
 Patrice Jeener, 33x25, 2003



Figure 15
Surface de Boy, François Apéry, 30x30x30, 2002

2.3 A few simple observations

Let us first consider parallel sections of the Boy surface (Figure 16). One can track their evolution on the following figures :

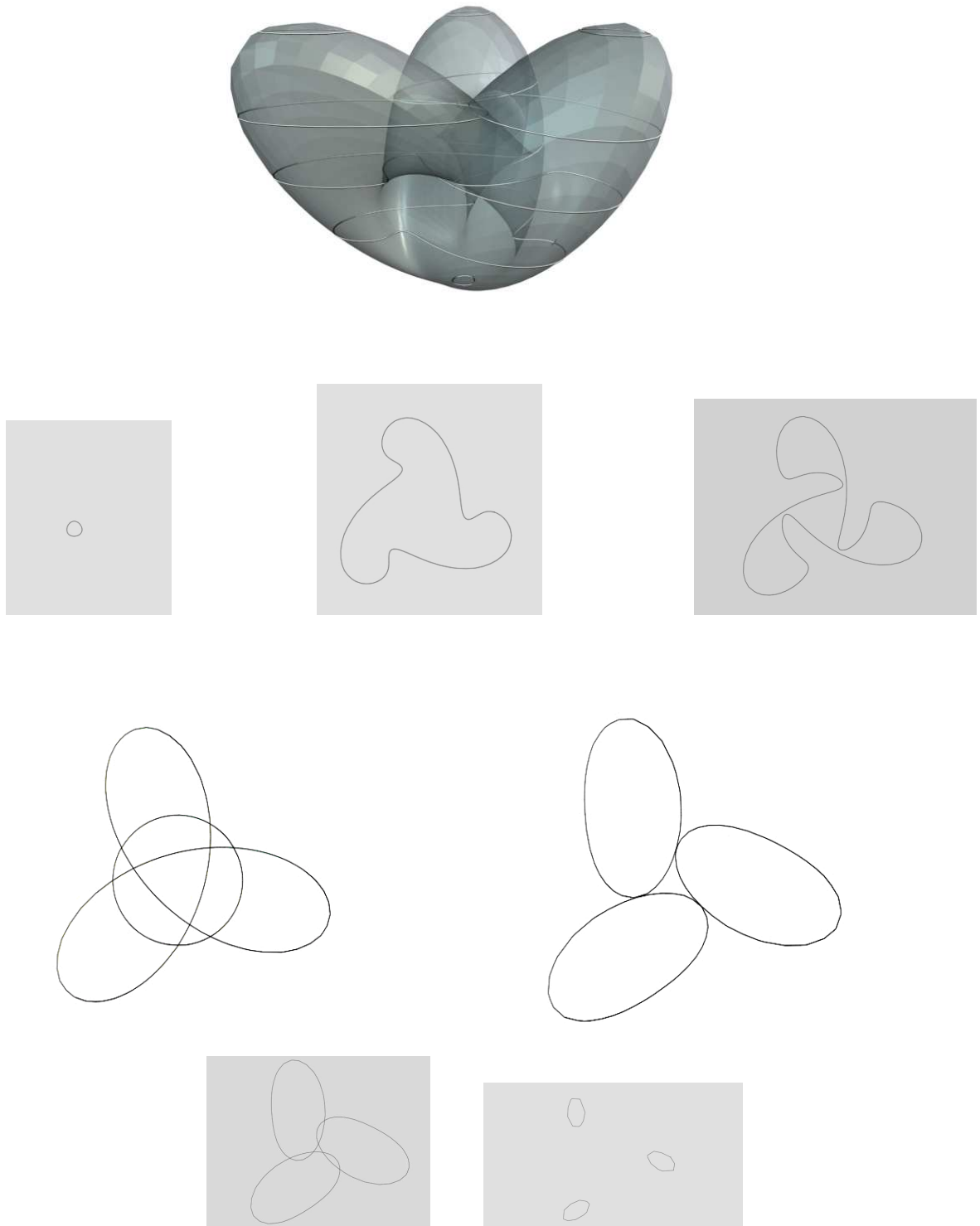


Figure 16 : Horizontal sections of the Boy surface, Christophe Delsart, 2008

Note that the Apéry's wire model lies at the singular "north pole". It allows us to interpret the Boy surface as a fiber space. Its basis is a convenient horizontal section of the surface : one can understand it as plane knot. There are at least two ways to look at the fibers : according to the first one, there are ellipses. Now, consider two infinitesimally close ellipses, they define an infinitesimal band : this band, which now plays the role of a fiber, is in fact a Möbius band having the north pole as a singularity where the twist happens ; it induces the non-orientation of the surface. The usual general theory of fiber spaces does not take into account the intersections of the basis and the fiber. But this intersection plays a role in the topology of the total space. Here each fiber, say each ellipse, intersects the basis in two points ; this property and the fact that the basis is a closed plane curve induces the presence of the self intersection curve in the surface.

The most simple and very singular example of such a situation is the 2-sphere : the basis is a circle, the fibers are also circles of the same size. This fiber space can be considered as a double covering of the 2-sphere. One can easily play with generalizations of this example, either to build n-spheres, or to construct other topological and geometrical 2 and 3-dimensional objects, by deforming the various circles and knots through different local homotopies. For instance, the fiber can be a trefoil, the basis might be an ellipse, and the fiber the apparent contour of a human face ; then we could look at this object from different points of views... .

The possible symmetry group of the basis induces the same symmetry group on the fiber space : the sphere, the Boy surfaces and the Morin surface are typical examples. These fiber spaces having some local sections with singularities belong to the category of what have been named "mountains".

3 The stained glass folly, a short description

3.1 The structure of the folly

Some well-chosen ellipses of the Boy surface, materialized in chrome steel, constitute the main part of the frame of the folly. This one has two parts : one which is visible from the outside, and the other one which can only be visible from the inside of the folly. Its floor lies on a perpendicular plane to the axis, located at approximately one third of the height of the construction, starting from the singular north pole. This floor is transparent so that one can see the lower part of the construction from the inside of the folly. This lower part lies inside a cylinder plastered with broken mirrors which reflect the internal lights illuminating the frame.

A few Möbius bands are materialized with stained glasses. By looking up and down, one can entirely see them from the inside of the folly. The outside part of the folly has for sure three pieces, but only two of them will serve as rooms for visitors. A series of vertical arches in the same material as the frame will cover the entrance of the two symmetric rooms. Here are a few preliminary shots of the folly (Figure 17), without any decoration, giving an idea of its structure.

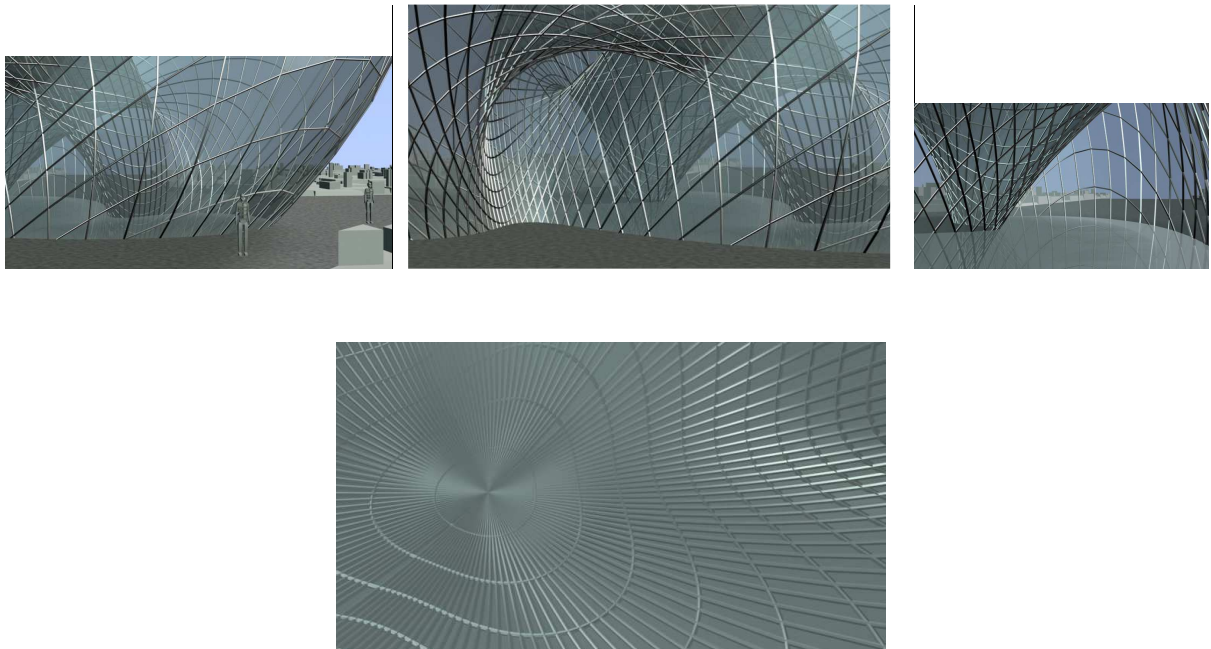
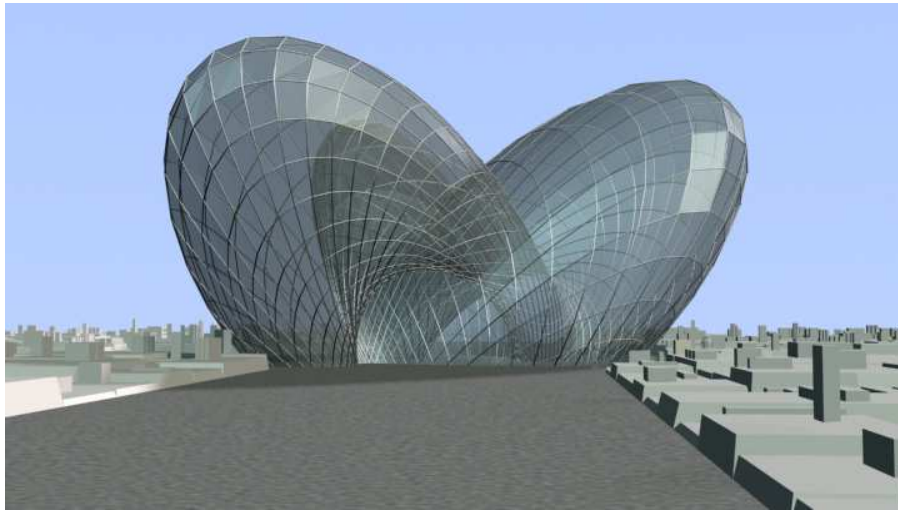


Figure 17
Preliminary images of the stained glass folly, Christophe Delsart and Yvan Ngnodjom, 2008

Convenient transversal rods to the elliptic frame will define panels and allow us to cover the outside part with glasses and mirrors. The elliptic curve will be replaced by a piecewise linear approximation. Then some of these panels will be partly stained glass windows showing either tilings or knots and braids with symmetries. LEDs inserted in some panels will follow curves from knots : their representation in space will be improved using the fact that the angle between two consecutive panels of glass is not flat.

Beams of search-lights with convenient colours will illuminate the folly, and four sculptures placed around as well.

3.2 The four sculptures

These sculptures should relate to Boy surface, by looking at it no more as a geometrical object but rather as a topological object. This surface has three visible lobes, it has one interior singular point, four exterior singular points : down, the “north pole”, and outside the submits of the three lobes. These singular points can be viewed as the centre and the vertices of a tetrahedron and the boundary of the Boy surface as a smooth deformation of a 2-sphere.

The projection of the tetrahedron on the ground can be done in such a way that it is an equilateral triangle.

A tetrahedron will be placed in the centre of this triangle. Each of its visible faces will be divided into three horizontal parts lying one upon each other. In each one, one or several knots linked together will be inserted. They can be made with different materials. Some of them will be screwed so that they can be taken off and replaced by others.

A sculpture conceived by Philippe Charbonneau will be placed in a back vertex of the triangle. It is a transparent ball made of resin or of glass, containing inside a part of a coppery ruled surface of degree 3. The intersection of this surface with the sphere is a double eight knot, that is three loops, the loop in the middle having one point in common with the left one, and an other point in common with the right one - this knot is the symbol of a Lens space $L(5,1)$. Here is a model of this sculpture (Figure 18) :



Figure 18 : Ruled surface inside a sphere, Philippe Charbonneau, 9, 1987

A third beautiful shining sculpture (Figure 19), made by Philippe Rips, shows a knot-network made with triangular rods. Such a shape was created maybe first by Alan Holden ([10], p. 125 and p.182 of the English edition, figures 171 and 243 of the French edition). I am indebted Jeremy Gray of the knowledge of this fact. Indeed the two realizations were made quite independently.

The general starting idea was to try to relate an object to some knots that may characterize it. At the last ISAMA Conference in Valencia, I learned about the works of Dmitri Kozlov [13]. My own approach was to begin with polyhedrons. A preliminary study should be to know which knots can be made from a given polyhedron considering not only its exterior edges, but also its interior edges

In the present case, the idea was to look for simple knots from some of the internal edges of a dodecahedron. The sculpture will be placed in the other back vertex of the triangle.

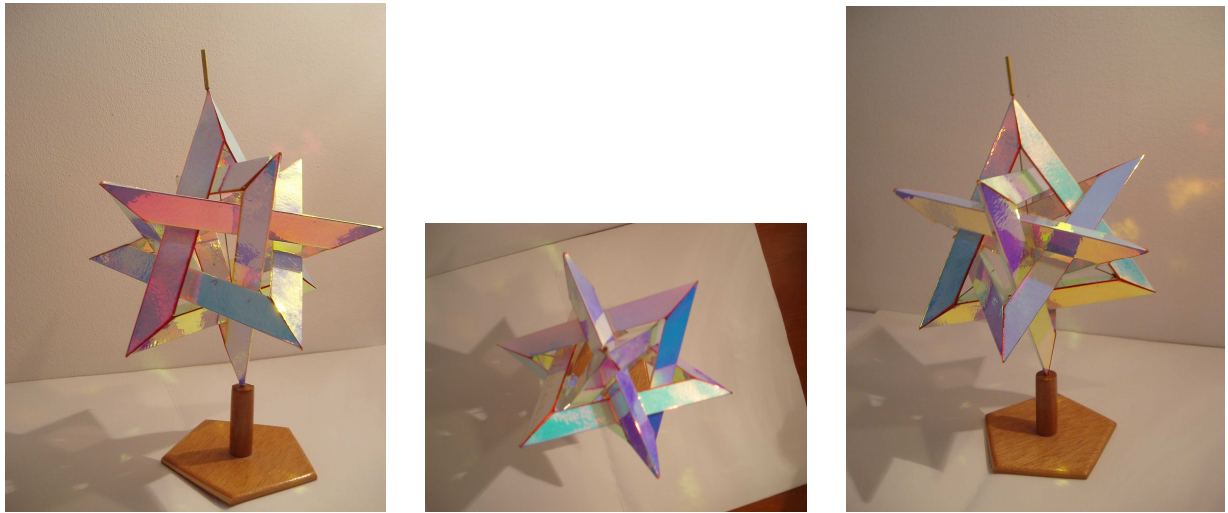


Figure 19 : Holden-Rips 4- Link, Philippe Rips, 15x15x 15, 2007

The last sculpture, again made by Philippe Rips, will be placed in the front vertex of the triangle. It needs some preliminary comments.

Along the previous program, after considering trivial knots, the following step is to look for the simplest non trivial knots, the trefoils then more generally the n -foils. Cellular decompositions of polyhedrons should be used in a further step.

As we have already seen, trefoils can be attached with frames in our space, which do not need to be orthogonal, in which case they are represented by three rods in general position. So that when one looks for trefoils created from the edges of a polyhedron, one has to look at trios of edges defining frames.

Philippe Rips, who first worked on tensegrity and tepees then on some classical objects, got interested in the problem.

Given its complexity, he chose to work on the truncated tetrahedron : note immediately that it looks a little bit like the boundary of a truncated 3 Boy surface.

Here is this truncated polyhedron (Figure 20), with a circular hole in each face in order to show some convenient internal edges.

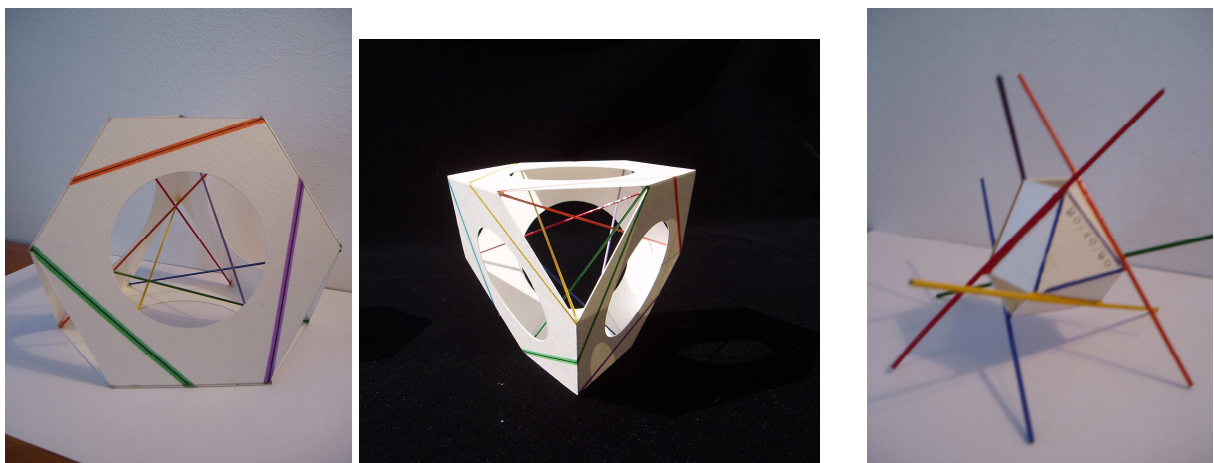


Figure 20 : Philippe Rips
Two internal views of a truncated tetrahedron, up and down, 2008

Figure 21 : Philippe Rips
The heart of the tetrahedron, 2008

As one can see better from the figure 21, by a circular permutation, we get five internal edges from one first, joining a vertex of the polyhedron to an opposite one. We get three couples of edges $A = (a, a')$, $B = (b, b')$, $C = (c, c')$, two edges of any couple being more or less parallel, so that the edges of any triple (α, β, γ) such that $\alpha \in A$, etc, are in general position. Starting with the triple (a, b, c) and, in order to illustrate a self-intersection, imposing that any letter appears twice, we get the four triples of frames :

$$(a, b, c), (a, b', c'), (b, c', a'), (c, a', b')$$

The code of colours is here :

$a = \text{blue}$, $b = \text{yellow}$, $c = \text{red}$, given rise to a trefoil coded in white.

$a' = \text{green}$, $b' = \text{orange}$, $c' = \text{violet}$, so that (a, b', c') (respectively (b, c', a') , (c, a', b')) gives rise to a trefoil coded in blue (respectively in yellow and red).

We can see now a few pictures of the set of the four trefoils associated with the truncated polyhedron (Figures 22 and 23) :

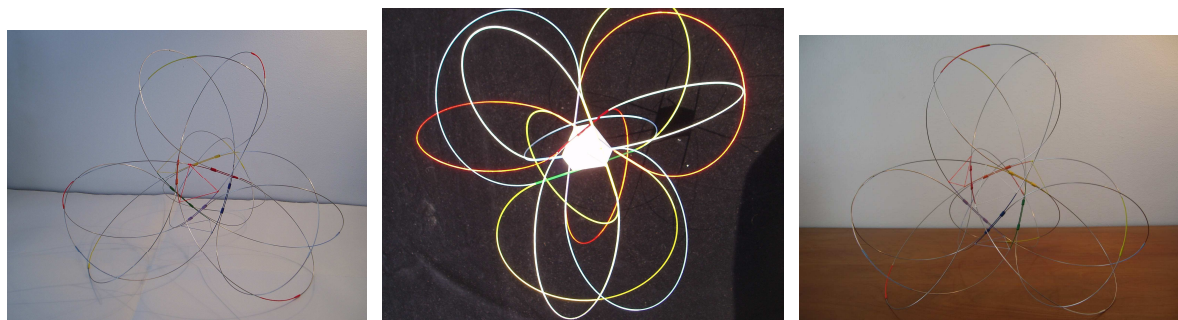


Figure 22
Philippe Rips, The Treasure of the Tetrahedron, 30 x30x 30, 2008
(middle : view from the top, left and right : front view)

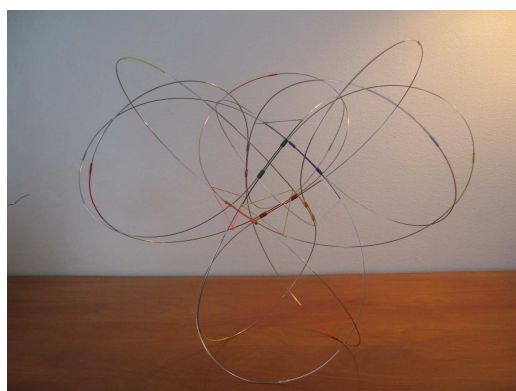


Figure 23
Philippe Rips, The Treasure of the Tetrahedron, 2008
(up or down)

As we know, Philippe Rips likes to produce flexible objects ; his home will resist any kind of seism.



Figure 24
Philippe Rips, Rips home, 2008

The man sitting is a mathematician¹ thinking about the open problem of finding the equations of the trefoils, (right, left, linked) related to a Boy surface.

Acknowledgements : Cyril Vachez put me in contact with Christophe Delsart and Yvan Ngnodjom, two students, first ignoring mathematics, from the Institut Multimedia of the Pôle Universitaire Léonard de Vinci. François Apéry gave them the equations of the 3-Boy surface so that Christophe and Yvan succeeded in visualizing the folly. I am happy to thank Cyril, Christophe, Yvan, François and Philippe Rips who has built such an interesting home. It is a pleasure to thank Nat Friedman for his linguistics improvements.

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¹ indeed a copy of a very old Romanian sculpture

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